COSMOLOGICAL VARIATION OF THE SCALAR FIELD AND ISOCHRONAL GEODESICS IN NON-COMPACTIFIED KALUZA THEORY

L.L. Williams
Konfluence Research Institute

ABSTRACT

Classical, non-compactified five-dimensional Kaluza theory is revisited. It is shown that the standard field equations imply that the scalar field characteristic of this theory varies only on cosmological scales. The constancy of the scalar field is therefore a valid approximation for the equations of motion on terrestrial scales. Previous work has suggested the scalar field can be identified with the gravitational constant. If so, then the gravitational constant is seen to be a function of the cosmological electromagnetic energy density.

Within the non-compactified framework, the cylinder condition of Kaluza theory is viewed as an approximation to the dependence of the fields on the fifth coordinate. The implications obtained in previous work of the cylinder condition are summarized: that electric charge arises from motion along the fifth coordinate, and that the gravitational constant provides a cosmic charge-to-mass ratio.

For motion on terrestrial scales, and under the approximation of the cylinder condition, the Kaluza theory is shown to admit isochronal geodesics, describing motion through space at constant time, for charged particles moving in magnetic fields.

I. INTRODUCTION

The five-dimensional Kaluza theory [6] provides an elegant unification of general relativity and classical electrodynamics. The four-dimensional Einstein equations with electromagnetic source terms, and the vacuum Maxwell equations, are both obtained from a set of five-dimensional vacuum Einstein equations. Likewise, the Lorentz force law obtains from a five-dimensional geodesic hypothesis. This unification comes at the cost of the ad hoc assumption known as the cylinder condition: that no fields depend on the fifth coordinate.

Starting with Klein [7,8], much of the subsequent work in what has come to be known as Kaluza-Klein theory has attempted to ascribe the cylinder condition to compactification of the fifth dimension down to unobservable microscopic scales. This concept of compactification has been a feature of subsequent higher-dimensional Kaluza-type unification theories. Another approach to the cylinder condition has made use of projective theories, in which the fifth dimension appears as a mathematical artifice derived from the usual four dimensions.

In more recent work, Overduin & Wesson [10] have revisited the compactified and projective approaches and compared them to Kaluza's original approach of taking the fifth dimension at face value, which Overduin & Wesson call the non-compactified approach. The cylindricity is relaxed in principle, to be interpreted merely as an approximation. While the compactified approach has been standard among workers in higher-dimensional relativity, Overduin & Wesson find no observational evidence to prefer a compatified or projective interpretation over a non-compactified one.

This paper considers five-dimensional Kaluza theory from a non-compactified viewpoint: that the fifth dimension is macroscopic, and the apparent independence of the fields from the fifth coordinate is only an approximation. While the underlying reason for this approximation remains undescribed, proceeding with a consideration of the properties of a macroscopic fifth dimension yields some results of interest.

Specifically, this paper focuses on implications of a macroscopic fifth dimension for the equations of motion. While [2] does make mention of how the fifth component of proper velocity of a particle reflects its charge-to-mass ratio, very little of the Kaluza literature before [10] focuses on the equations of motion. Therefore this paper begins with an enunciation of the properties of the fifth component of the proper velocity, and of its theoretical context. One finds that that the gravitational constant provides a cosmic charge-to-mass ratio, and provides a conversion constant between units of length and units characterizing the fifth dimension. Electric charge is seen to be only the fifth-dimensional projection of an energy-momentum-charge five-vector and therefore not strictly conserved, implying the existence of a rest charge. The units of the fifth dimension are characterized.

The scalar field inherent in the five-dimensional theory is revisited, and it is shown how its scale of variation decouples from the terrestrial variational scales of the other fields. Making the Brans-Dicke identification of the scalar field with the gravitational constant, one is lead to the result that the gravitational constant depends on the cosmological electromagnetic energy density.

Working within the cylinder approximation, and ignoring variation of the scalar field on terrerstrial scales, isochronal geodesics, which describe motion in three-space at constant time, obtain from motion of charged particles in magnetic fields. These equations of motion through space at constant time are described.

II. GEODESIC HYPOTHESIS IN FIVE DIMENSIONS

The following notation is adopted. Five-dimensional (5D) tensors are indicated with a tilde to distinguish them from four-dimensional (4D) ones. The time coordinate is x^0, and the spatial coordinates x^1, x^2, x^3. The fifth coordinate is x^5. Summation is implied on repeated pairs of covariant and contravariant indices. Roman indices range over all five coordinates, and greek indices over the usual four coordinates of space and time. Partial derivatives $\partial/\partial x^a$ are abbreviated ∂_a.

In a space described by the 5D metric $\widetilde{g}_{ab}$, particles of matter move along 5D paths described in terms of a constant b, and a 5D proper time $\widetilde{\tau}$:

$$\widetilde{g}_{ab} dx^a dx^b \equiv b^2 d\widetilde{\tau}^2, \tag{1}$$

which constrains the 5D length of the 5D proper velocity, $\widetilde{U}^a \equiv dx^a/d\widetilde{\tau}$. Previous results [2,3] show that the signature of the fifth dimension in the metric must be spacelike. The entire space x^a is hyperbolic, but the subspace of x^1, x^2, x^3, x^5 is Euclidean.

The 5D equations of motion obtain from the 5D geodesic hypothesis:

$$\widetilde{U}^b \widetilde{\nabla}_b \widetilde{U}^a \equiv \frac{d\widetilde{U}^a}{d\widetilde{\tau}} + \widetilde{\Gamma}^a_{bc} \widetilde{U}^b \widetilde{U}^c = 0, \tag{2}$$

where $\widetilde{\nabla}_a$ is the 5D covariant derivative. $\widetilde{\Gamma}^a_{bc}$ is the 5D affine connection, which has the standard relation to the metric:

$$2\widetilde{\Gamma}^a_{bc} = \widetilde{g}^{ad} \left[\partial_c \widetilde{g}_{bd} + \partial_b \widetilde{g}_{cd} - \partial_d \widetilde{g}_{bc} \right]. \tag{3}$$

The modified 4D equations of motion are given by the 4D spacetime components of (2):

$$\frac{d\widetilde{U}^\nu}{d\widetilde{\tau}} + \widetilde{\Gamma}^\nu_{55} (\widetilde{U}^5)^2 + 2\widetilde{\Gamma}^\nu_{5\mu} \widetilde{U}^5 \widetilde{U}^\mu + \widetilde{\Gamma}^\nu_{\alpha\beta} \widetilde{U}^\alpha \widetilde{U}^\beta = 0. \tag{4}$$

Expressed in terms of the 4D proper time τ defined by $g_{\mu\nu} dx^\mu dx^\nu = c^2 d\tau^2$, where $g_{\mu\nu}$ is the 4D metric and c is the speed of light, (4) can be rewritten in terms of the 4D proper velocity $U^\nu \equiv dx^\nu/d\tau$:

$$\frac{dU^\nu}{d\tau} + \widetilde{\Gamma}^\nu_{55} (U^5)^2 + 2\widetilde{\Gamma}^\nu_{5\mu} U^5 U^\mu + \widetilde{\Gamma}^\nu_{\alpha\beta} U^\alpha U^\beta + \left(\frac{d\widetilde{\tau}}{d\tau} \right)^2 U^\nu \frac{d^2\tau}{d\widetilde{\tau}^2} = 0. \tag{5}$$

Equation (5) is the generalized spacetime equation of motion.

III. THE KALUZA METRIC AND THE INTERPRETATION OF CHARGE

The Kaluza metric $\widetilde{g}_{ab}$ is completely determined by demanding $\widetilde{\Gamma}^\nu_{5\mu}$ take a form such that the term in (5) linear in U^ν corresponds to the Lorentz force vector (in Gaussian units),

$$2\widetilde{\Gamma}^\nu_{5\mu} U^5 U^\mu \Longrightarrow \frac{q}{m_0 c} F_{\alpha\beta} U^\beta g^{\alpha\nu}, \tag{6}$$

where $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$, A^α is the electromagnetic four-vector potential, m_0 is particle rest mass, and q is particle charge. There are in general other terms besides the Lorentz term which may arise from $\widetilde{\Gamma}^\nu_{5\mu}$, but correspondence with 4D physics requires the Lorentz term to be among them.

The five-dimensional metric $\widetilde{g}_{ab}$ is therefore given in terms of $g_{\mu\nu}$ and A^μ:

$$\begin{aligned} \widetilde{g}_{\mu\nu} &= g_{\mu\nu} + k^2\phi A_\mu A_\nu \qquad \widetilde{g}^{\mu\nu} = g^{\mu\nu} \\ \widetilde{g}_{5\nu} &= k\phi A_\nu \qquad \widetilde{g}^{5\mu} = -kA^\mu \\ \widetilde{g}_{55} &\equiv \phi \qquad \widetilde{g}^{55} = 1 + k^2A^2 \end{aligned} \tag{7}$$

where $A^2 \equiv A_\alpha A^\alpha = g_{\alpha\beta}A^\alpha A^\beta$, $\widetilde{g}_{ab}\widetilde{g}^{bc} = \delta^c_a$, and k is a constant. The price of unifying $g_{\mu\nu}$ and A^μ within the framework of a 5D metric is the necessary introduction of the scalar field ϕ.

The 5D metric (7) was studied in a series of classic papers including Kaluza [6], Klein [7], and Thiry [9]. Closely related papers included those by Klein [8], Jordan [5], and Brans & Dicke [4]. Overduin & Wesson [10], Bergmann [3], and Bargmann [2] provide reviews. Applequist, Chodos, and Freund [1] provide translations of [6] and [7] and some historical review.

In addition to assuming the form of $\widetilde{g}_{ab}$ to recover the Lorentz force, it is also necessary to identify the proper velocity along the fifth coordinate with electric charge:

$$U^5 \equiv \frac{dx^5}{d\tau} = \frac{q/m_0c}{k}. \tag{8}$$

The 5D metric (7) is rich enough in structure that general relativity and electrodynamics, along with significant coupling terms, can be recovered in the absence of any variation of $\widetilde{g}_{ab}$ with x^5. The simplest form of Kaluza theory, and the one originally put forth by Kaluza, therefore assumes $\partial\widetilde{g}_{ab}/\partial x^5 = 0$, the relation historically known as the cylinder condition. In the non-compactified approach, this condition is considered only an approximation, and the fields may in general have some dependence, albeit weak, on the fifth coordinate. The subsequent development here makes the approximation of cylindricity in deriving the equations of motion. Overduin & Wesson explore the implications of relaxing the cylinder condition.

IV. THE FIELD EQUATIONS CONSTRAIN k AND ϕ

A standard result of Kaluza theory is that the constant k is fixed by correspondence of the 5D vacuum Einstein equations to the 4D Einstein equations.

The 5D Einstein equations,

$$\widetilde{G}_{ab} \equiv \widetilde{R}_{ab} - \frac{1}{2}\widetilde{g}_{ab}\widetilde{R} = 0, \tag{9}$$

where $\widetilde{R}_{ab}$ is the 5D Ricci tensor, and $\widetilde{R}$ is the 5D scalar curvature, are decomposed within the cylinder approximation into the familiar 4D components [9]:

$$\widetilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{k^2\phi}{2}T^{EM}_{\mu\nu} - T_{\mu\nu}(\phi), \tag{10}$$

$$\nabla^\mu F_{\mu\nu} = -3\phi^{-1/2}g^{\mu\alpha}\frac{\partial\phi^{1/2}}{\partial x^\alpha}F_{\mu\nu}, \tag{11}$$

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi^{1/2} = \frac{k^2\phi^{3/2}}{4}F_{\alpha\beta}F^{\alpha\beta}, \tag{12}$$

where ∇_α is the covariant 4D derivative, $T^{EM}_{\mu\nu}$ is the electromagnetic stress tensor, and $T_{\mu\nu}(\phi)$ is a function of ϕ.

The spacetime components of (10) include source terms in $g_{\mu\nu}$ from both the electromagnetic field and ϕ. Furthermore, k is determined by demanding correspondence between (10) and the 4D field equations with an electromagnetic source:

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T^{EM}_{\mu\nu}, \tag{13}$$

so (10) and (13) imply

$$\phi k^2 = \frac{16\pi G}{c^4}. \tag{14}$$

G is the gravitational constant, and its units are $l^3m^{-1}t^{-2}$, while those of the vector potential A are $m^{1/2}l^{1/2}t^{-1}$, where m denotes a mass unit, l a length unit, and t a time unit. Therefore $k^2A^2 \sim GA^2/c^4$ is the dimensionless number that expresses the strength of the coupling corrections to the 4-D theory.

k^2 is a very small number, and quantifies weak couplings to electromagnetic fields. For a neutron star magnetic field of strength 10^{12} G and lengthscale 10 km, GA^2/c^4 is of order 10^{-13}. For galactic electromagnetic field energy densities of 1 eV cm^{-3} and length scale 1 parsec, GA^2/c^4 is of order 10^{-32}. The metric terms linear in kA are strong enough to produce the Lorentz force because U^5 is inversely proportional to k.

The scalar field must be assimilated into a coherent framework of understanding that still provides the 4D classical limits. In fact, as summarized by Applequist et al.[1], the significance of ϕ in the field equations was not fully appreciated until some two decades after Kaluza's paper. Although Kaluza originally assumed ϕ constant, it was not realized until much later the implied constraint (12) on the electromagnetic field. However, the magnitude of k implies constant ϕ is a valid assumption on terrestrial scales.

The field equations for ϕ (12) and the expression for k (14) imply a scale of variation for ϕ given by $l_\phi \sim (k^2 e_{EM})^{-1/2} \sim c^2/(G e_{EM})^{1/2}$, where e_{EM} is electromagnetic energy density. For an electromagnetic energy density similar to the observed average galactic value of 1 eV cm^{-3}, the magnitude of k implies ϕ varies on a lengthscale similar to the radius of the universe. The scale of variation of ϕ arising from a neutron-star magnetic field of 10^{12} Gauss would be of order 1 astrononmical unit. To achive variations in ϕ on a scale of kilometers would require energy densities of order 10^{40} erg cm^{-3} or magnetic field strengths of order 10^{20} G.

Variation of ϕ is therefore of interest only on cosmological scales, and not of interest for the equations of motion of terrestrial objects. So ϕ can be approximated as a constant in the equations of motion of terrestrial objects, and the constraint (12) from assuming constant ϕ is understood to be only on the cosmological electromagnetic energy density.

As suggested by Bran & Dicke [4] and Jordan [5], and implied by (14), ϕ may be interpreted as the gravitational constant G. In this case, (12) implies the gravitational constant is a function of the cosmological electromagnetic energy density.

The constancy of ϕ on terrestrial scales, and the cylinder approximation, makes the relation (6) between the 5D connection $\widetilde{\Gamma}^{\nu}_{5\mu}$ in (5) and the Lorentz force exact. The approximation of constant ϕ, as for the cylinder approximation, greatly simplifies the 5D connections, and therefore the equations of motion. Even in this simplest of mathematical limits, there emerge non-trivial electromagnetic corrections to the equations of motion.

V. G AND THE NATURE OF x^5

Equations (8) and (14) imply the x^5 component of the proper velocity is

$$U^5 = c\,\frac{q/m_0}{4(\pi G)^{1/2}}. \tag{15}$$

The origin of electric charge finds its explanation in this theory as arising from "motion" along the 5th coordinate. Just as particle energy $\propto U^0$ and spatial momentum $\propto \mathbf{U}$ are frame-dependent, so is electric charge.

Just as the time coordinate $\Delta x^0 \equiv c\Delta t$ is normalized to units of length with the constant c, so may x^5 be decomposed into a 'natural' coordinate h and a constant α: $\Delta x^5 \equiv \alpha \Delta h$.

Now the question arises as to which of the physical quantities on the right hand side of (15) is associated with motion along the fifth coordinate. Even as G and c are understood to be constants independent of particle motion, (15) still does not provide a clear basis for distinguishing only electric charge as originating from motion along the fifth coordinate. It is the ratio of charge to rest mass which enters (15).

However, the assumption is taken here that rest mass is a parameter independent of electric charge, and it is electric charge only which is identified with the natural coordinate h: $q \equiv dh/d\tau$. The rest mass is pulled into the normalizing constant α such that $\alpha \equiv c/(16\pi G m_0^2)^{1/2}$. In this case, the natural coordinate h is seen to have units of charge·time.

The gravitational constant takes on new significance in light of (15). The Gaussian units of electric charge are $m^{1/2} l^{3/2} t^{-1}$, so $G^{1/2}$ provides a universal charge-to-mass ratio. The values of U^5 for elementary particles are astronomical: $10^{18}c$ for protons, and $10^{21}c$ for electrons.

This immediately creates an apparent problem in the equations of motion (5) because the term quadratic in U^5 would dominate the Lorentz and Einstein force terms. This apparent problem was noticed originally by Kaluza [6]. However, under the assumption of constant ϕ, and the implied constancy of the cosmological electromagnetic energy density, the term in (5) quadratic in U^5 vanishes.

VI. CYLINDER CONDITION AND VARIATION OF CHARGE

The approximation of the cylinder condition allows recourse to a simple equation of motion for $\widetilde{U}^5$. The equations of motion (2) can be written in a simple and completely general form for the covariant component of the 5-velocity $\widetilde{U}_a = \widetilde{g}_{ab}\widetilde{U}^b$. Because the metric commutes with the covariant derivative $\widetilde{\nabla}_a$, (2) and the antisymmetry in (3) imply:

$$\frac{d\widetilde{U}_a}{d\widetilde{\tau}} = \frac{1}{2}\widetilde{U}^b \widetilde{U}^c \frac{\partial \widetilde{g}_{bc}}{\partial x^a}. \tag{16}$$

This form of the equations of motion is in terms of a normal partial derivative instead of a covariant derivative, and expresses the conservation properties of the metric. There is a conserved quantity associated with each invariant coordinate.

The cylinder approximation therefore implies a conserved quantity along 5D worldlines:

$$\widetilde{U}_5 = \phi k A_\nu \widetilde{U}^\nu + \phi \widetilde{U}^5 = \left(\frac{d\tau}{d\widetilde{\tau}}\right) \phi (k A_\nu U^\nu + U^5) = \text{constant}. \tag{17}$$

Equations (17), (1), and (7) imply a simple relation between the 5D and 4D proper times:

$$\left(\frac{d\tau}{d\widetilde{\tau}}\right)^2 = b^2(c^2 + \widetilde{U}_5^2/\phi)^{-1} \Longrightarrow \text{constant}. \tag{18}$$

The constancy of ϕ on terrestrial scales implies the constancy of (18), and eliminates the final term in (5). $d\tau/d\widetilde{\tau}$ differs from 1 only for charged particles, or uncharged particles in electromagnetic fields: verify this.

Note that it is U^5 which is identified with electric charge in the equations of motion (5), while (17) shows that U^5 alone is not conserved. Charge is affected by motion in electromagnetic fields. With $\phi = -1$, equation (17) can be rewritten in conventional terms:

$$\frac{16\pi G}{c^3} m_0 A_\nu U^\nu - q = \text{constant}. \tag{19}$$

Electric charge is a frame-dependent quantity, and its variation with motion in electromagnetic fields is given by (19). For a relativistic proton with proper velocity $\sim 100c$, moving in a 10^6 Gauss magnetic field characterized by a lengthscale of 10 meters, the variation in charge is $\sim 10^{-30}$. Therefore, for elementary charged particles, the rest charge q_0 and its variation with motion q are approximately equal and constant, and cancel out of the energy-momentum-charge relations (16), preserving the standard 4D energy-momentum relations.

IX. MOTION AT CONSTANT TIME

The charge of particles at rest in the laboratory is interpreted within the 5D theory as motion along the 5th coordinate at constant spatial position. We would like to investigate whether x^5 can serve as the independent variable for spatial derivatives in the absence of variation in time. That is, we seek geodesics in the space of x^1, x^2, x^3, x^5 that lie on slices of constant time.

The constant b can be obtained from the 5D Minkowski limit, when $\widetilde{g}_{ab}$ is diagonal and constant. The 5D equivalence principle demands that we be able to tranform to a coordinate system characterized by the 5D Minkowski metric, where ϕ, as shown in previous work, must share the sign of the spatial dimensions. In the absence of electromagnetic or gravitational fields, and in the absence of spatial motion

$$\widetilde{g}_{ab}\widetilde{U}^a\widetilde{U}^b = \widetilde{g}_{ab}U^aU^b\left(\frac{d\tau}{d\widetilde{\tau}}\right)^2 = b^2 = c^2\left[1 - \frac{(q/m_0)^2}{16\pi G}\right]\left(\frac{d\tau}{d\widetilde{\tau}}\right)^2. \tag{20}$$

Consider a charged particle at rest. Its 5D proper velocity $\widetilde{U}^a = (dx^0/d\widetilde{\tau}, dx^5/d\widetilde{\tau})$, and $\widetilde{U}^2 = b^2$ is given by (20). Consider now motion in $(\mathbf{x}, x^5)$ for $dx^0 = 0$, where $\mathbf{x}$ is the spatial position vector. Then

$$\widetilde{U}^a = \left(0, \frac{d\mathbf{x}}{d\widetilde{\tau}}, \frac{dx^5}{d\widetilde{\tau}}\right) = \frac{d\tau}{d\widetilde{\tau}}\left(0, \frac{d\mathbf{x}}{d\tau}, \frac{dx^5}{d\tau}\right) = \frac{dh}{d\tau}\frac{d\tau}{d\widetilde{\tau}}\left(\frac{d\mathbf{x}}{dh}, \alpha\right), \tag{21}$$

and

$$\widetilde{U}^2 = -\left(\frac{d\tau}{d\tau}\right)^2\left(\frac{dh}{d\widetilde{\tau}}\right)^2\left[\left(\frac{d\mathbf{x}}{dh}\right)^2 + \alpha^2\right] \equiv -q_5^2\left[v_5^2 + \alpha^2\right]\left(\frac{d\tau}{d\widetilde{\tau}}\right)^2. \tag{22}$$

Now demand that the charged particle at rest somehow changes its motion to that described by (21). Then $\widetilde{U}^2$ as described by (20) must equate to that described by (22), thereby determining q_5 on geodesics of $dx^0 = 0$:

$$q_5^2 = \frac{q_0^2\alpha^2 - c^2}{v_5^2 + \alpha^2} \tag{23}$$

The cylinder condition (17) implies

$$\widetilde{U}_5 = -q_5\left[k\mathbf{A}\cdot\mathbf{v}_5 + 1\right] = \text{constant} \tag{24}$$

which, with (23), describes the motion through space at constant time for particles of rest mass m_0 and electric rest charge q_0. These geodesics at constant time apparently depend on an applied vector potential $\mathbf{A}$, with associated magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$.

X. CONCLUSIONS

While early treatments of Kaluza theory ignored variation of the scalar field, and thereby imposed an artificial constraint on the electromagnetic energy density, it turns out that the field equations imply that the scalar field varies only on cosmological scales. Therefore constancy of the scalar field is a valid approximation for motion on terrestrial scales. If the scalar field is identified with the gravitational constant, it implies that the gravitational constant is a function of the cosmological electromagnetic energy density.

For motion on terrestrial scales, and under the approximation of the cylinder condition, non-compactified Kaluza theory admits isochronal geodesics for the motion of charged particles in magnetic fields.

XI. REFERENCES

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