ABSTRACT

Classical 5D Kaluza theory predicts

- an electrodynamic antigravity which does not couple to charge
- the gravitational constant is a charge-to-mass ratio. It is also the coupling constant between gravity and EM
- G is analagous to c in that both provide dimensional conversion to length units
- G/c^4 appears to be the Brans Dicke scalar
- equations in principle allow motion at constant time
- the equations are applicable over a broad range of lengthscales, and variations in the scalar field are not terrestrial

1. Review of 4D Equations of Motion

A cornerstone of relativity is the geodesic hypothesis: that there exists a 4D spacetime coordinate system ξ^μ in which the equation of motion for a particle moving in a gravitational field describes a straight line:

$$\frac{d^2\xi^\mu}{d\tau^2} = 0 \tag{1}$$

$$c^2 d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{2}$$

In other words, the flat Minkowskian metric of special relativity is locally and instantaneously, accurate everywhere. One may then choose to find ξ^μ in lab coordinates x^μ:

$$\frac{d\xi^\mu}{d\tau} = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}$$

And,

$$\frac{\partial x^\delta}{\partial \xi^\alpha}\frac{d^2\xi^\alpha}{d\tau^2} = \frac{d^2x^\delta}{d\tau^2} + \frac{\partial x^\delta}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\beta \partial x^\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$$

The equation of motion of gravitational free fall is then:

$$\frac{d^2x^\delta}{d\tau^2} + \Gamma^\delta_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \equiv U^\alpha \nabla_\alpha U^\delta = 0 \tag{3}$$

where

$$\Gamma^\alpha_{\beta\gamma} \equiv \frac{\partial x^\alpha}{\partial \xi^\sigma}\frac{\partial^2 \xi^\sigma}{\partial x^\beta \partial x^\gamma} = \frac{1}{2} g^{\alpha\delta}\left\{\frac{\partial g_{\beta\delta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\delta}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right\} \tag{4}$$

and we have defined the 4D covariant derivative ∇_μ.

The generalized metric $g_{\alpha\beta}$ is related to the coordinate transformation of the line element:

$$c^2 d^2\tau \equiv \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = \eta_{\gamma\delta}\frac{\partial \xi^\gamma}{\partial x^\alpha}\frac{\partial \xi^\delta}{\partial x^\beta} dx^\alpha dx^\beta \equiv g_{\alpha\beta} dx^\alpha dx^\beta$$

so that

$$g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = c^2 \tag{5}$$

which complements (3) to describe the motion of a particle moving freely under the force of gravity.

In the presence of electromagnetic forces, the relativistic equations of motion take the form

$$\frac{dU^\delta}{d\tau} + \Gamma^\delta_{\beta\gamma} U^\beta U^\gamma = \frac{q}{mc} F_{\alpha\gamma} U^\gamma g^{\alpha\delta} \tag{6}$$

for objects of rest mass m, charge q, and 4-velocity U^μ, where

$$U^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} \equiv \gamma(c, \mathbf{v}) \tag{7}$$

Equations (5) and (7) imply the 4-velocity has invariant length and allows one to solve for γ under the assumption that $\gamma(\mathbf{v} = 0) = 1$.

$$\gamma^2(c^2 - v^2) = c^2$$

2. 5D Equations of Motion

Now apply Kaluza's hypothesis to the geodesic equation. Can we recover the electromagnetic equations of motion by extending the geodesic hypothesis to five dimensions? It turns out the answer is yes. Also, considering the equations of motion determines the extra-dimensional extensions to the 4-D metric. A free constant is fixed by identifying a component of the 5-D Einstein equations with the Maxwell equations. All in all, makes for quite a tidy bundle of physics.

In the following, greek indices are reserved for the 4 spacetime dimensions. Roman indices range over all 5 dimensions. 5-D quantities will be indicated with a tilde, e.g., $\widetilde{g}_{\mu\nu} \neq g_{\mu\nu}$. The time dimension is denoted 0, spatial is 1,2,3, and 5 for the 5th dimension.

Extending relativity to 5D demands an assumption about the 5D Minkowski metric, $\widetilde{\eta}_{ab}$. Assume the 5D extensions to the flat spacetime metric:

$$\widetilde{\eta}_{00} = +1;\ \widetilde{\eta}_{11} = \widetilde{\eta}_{22} = \widetilde{\eta}_{33} = -1;\ \widetilde{\eta}_{55} = \mathrm{e}^{i\theta}$$

where an arbitrary complex metric coefficient of unit modulus is allowed in the fifth diagonal element.

Assuming an invariant interval $b^2 d\widetilde{\tau}^2$, the 5-D analog of the geodesic hypothesis, (1) and (2):

$$\frac{d^2\xi^a}{d\widetilde{\tau}^2} = 0 \tag{8}$$

$$b^2 d\widetilde{\tau}^2 = \widetilde{\eta}_{ab} d\xi^a d\xi^b \tag{9}$$

lead by the same mathematics to the 5-D equations of motion:

$$\frac{d\widetilde{U}^a}{d\widetilde{\tau}} + \widetilde{\Gamma}^a_{bc}\widetilde{U}^b\widetilde{U}^c = 0 \equiv \widetilde{U}^b\widetilde{\nabla}_b\widetilde{U}^a \tag{9}$$

$$\widetilde{g}_{ab}\frac{dx^a}{d\widetilde{\tau}}\frac{dx^b}{d\widetilde{\tau}} = b^2 \tag{10}$$

where the constant in (10) is calculated below. Assume that 4 of the 5 dimensions are the regular spacetime dimensions of relativity: $x^a = (x^\mu, x^5) = (ct, \mathbf{x}, sh)$, where the constant s converts the 5th dimension units to length units, just as c does for time: $x^0 = ct$ and $x^5 = sh$. Tildes also differentiate 4D and 5D derivatives such as $\widetilde{U}_\nu \neq U_\nu$. The 5D connections are as given in (4) with summation over 5 indices.

Now define the 5-velocity vector:

$$\widetilde{U}^a \equiv \frac{dx^a}{d\widetilde{\tau}} = (\widetilde{U}^\mu, \widetilde{U}^5) = \left(\frac{dt}{d\widetilde{\tau}}\right)\left(\frac{dx^\mu}{dt}, \frac{dx^5}{dt}\right) \equiv \widetilde{\gamma}\left(c, \mathbf{v}, s\dot{h}\right) = \frac{d\tau}{d\widetilde{\tau}}\left(U^\mu, U^5\right) \tag{11}$$

Then the equations of motion through the 5 dimensions, factored for identification with 4D theory, are:

$$\begin{aligned}\frac{d\widetilde{U}^\nu}{d\widetilde{\tau}} &+ \widetilde{\Gamma}^\nu_{55}(\widetilde{U}^5)^2 + 2\widetilde{\Gamma}^\nu_{5\mu}\widetilde{U}^5\widetilde{U}^\mu + \widetilde{\Gamma}^\nu_{\alpha\beta}\widetilde{U}^\alpha\widetilde{U}^\beta = 0 \\ &= \left(\frac{d\tau}{d\widetilde{\tau}}\right)^2\left(\frac{dU^\nu}{d\tau} + \widetilde{\Gamma}^\nu_{55}(U^5)^2 + 2\widetilde{\Gamma}^\nu_{5\mu}U^5U^\mu + \widetilde{\Gamma}^\nu_{\alpha\beta}U^\alpha U^\beta\right) + U^\nu\frac{d^2\tau}{d\widetilde{\tau}^2}\end{aligned} \tag{12}$$

$$\frac{d\widetilde{U}^5}{d\widetilde{\tau}} + \widetilde{\Gamma}^5_{55}(\widetilde{U}^5)^2 + 2\widetilde{\Gamma}^5_{5\mu}\widetilde{U}^5\widetilde{U}^\mu + \widetilde{\Gamma}^5_{\alpha\beta}\widetilde{U}^\alpha\widetilde{U}^\beta = 0 \tag{13}$$

3. U^5, $\widetilde{g}_{ab}$ from 4D Lorentz limit

The 5D geodesic equations (12) involve a term linear in the four-velocity that invites identification with the electromagnetic term in (6). To make this correspondence would require

$$2\widetilde{\Gamma}^\nu_{5\mu}U^5 = -\frac{q}{mc}F_{\alpha\mu}g^{\alpha\nu} \tag{14}$$

where

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{15}$$

(14) expands to:

$$U^5\left[\widetilde{g}^{\alpha\nu}(\partial_\mu\widetilde{g}_{5\alpha} - \partial_\alpha\widetilde{g}_{5\mu}) + \widetilde{g}^{\nu\alpha}\partial_5\widetilde{g}_{\mu\alpha} + \widetilde{g}^{\nu5}\partial_\mu\widetilde{g}_{55}\right] = \frac{q}{mc}g^{\alpha\nu}\ (\partial_\mu A_\alpha - \partial_\alpha A_\mu) \tag{16}$$

Relaxing the equalities (14) and (16) for the moment, we would at least recover (6) among other terms if

$$\widetilde{g}^{\alpha\nu} = g^{\alpha\nu} \tag{17}$$

$$\widetilde{g}_{5\alpha} = kA_\alpha \tag{18}$$

$$U^5 = \frac{dx^5}{d\tau} = \frac{q}{kmc} \tag{19}$$

where k is a constant that will ultimately be determined by the Einstein-Maxwell equations. The coefficient in (17) must be unity to preserve the limit $\widetilde{g}^{\mu\nu} \Longrightarrow g^{\mu\nu}$ when $A = 0$.

In terms of $\widetilde{g}_{55} \equiv \phi$, (17) and (18) are sufficient to specify $\widetilde{g}_{ab}$ and its inverse:

$$\begin{aligned}
\widetilde{g}_{\mu\nu} &= g_{\mu\nu} + k^2 A_\mu A_\nu/\phi \\
\widetilde{g}_{5\nu} &= kA_\nu \\
\widetilde{g}_{55} &\equiv \phi \\
\widetilde{g}^{\mu\nu} &= g^{\mu\nu} \\
\widetilde{g}^{5\mu} &= -kA^\mu/\phi \\
\widetilde{g}^{55} &= \frac{1}{\phi} + \frac{k^2 A^2}{\phi^2}
\end{aligned} \tag{20}$$

where $A^2 \equiv A_\alpha A^\alpha = g_{\alpha\beta} A^\alpha A^\beta$. Now, $\widetilde{g}_{ab}\widetilde{g}^{bc} = \delta^c_a$. $g_{\mu\nu}$ is the object used to raise and lower indices on 4-D objects. However, it is $\widetilde{g}_{ab}$ which raises and lowers indices on 5-D objects. This is the classic metric of Kaluza (1921), Klein (1926), Bargmann (1957), Thiry (1948), and Wesson (1999).

4. Transformation properties of $\widetilde{g}_{ab}$

The metric (20) has suitable transformation properties, expressed in terms of coordinates x^a and x'^a:

$$\widetilde{g}'_{\alpha\beta} - \widetilde{g}'_{\alpha 5}\widetilde{g}'_{\beta 5} = \left(\widetilde{g}_{\mu\nu} - \widetilde{g}_{\mu 5}\widetilde{g}_{\nu 5}\right) \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}$$

Under the assumption that the 5D transformation properties G_5 are related to the normal 4D group G_4 such that $G_5 = G_4 S_1$, with

$$x'^\mu = f(x^\mu); \qquad x'^5 = x^5 + g(x^\mu)$$

Then ϕ is independent of coordinate system:

$$\widetilde{g}'_{55} = \widetilde{g}_{55}$$

If the metric is independent of x^5, an assumption taken here, it is independent of x'^5:

$$\frac{\partial \widetilde{g}_{ab}}{\partial x^5} = 0 = \frac{\partial \widetilde{g}_{ab}}{\partial x'^5}\frac{\partial x'^5}{\partial x^5} = \frac{\partial \widetilde{g}_{ab}}{\partial x'^5}$$

The component of the metric we have identified with the potential 4-vector transforms like a 4-vector to within the gradient of a scalar:

$$\widetilde{g}'_{\alpha 5} = \widetilde{g}_{55}\frac{\partial f}{\partial x^\alpha} + \widetilde{g}_{\mu 5}\frac{\partial x'^\mu}{\partial x^\alpha}$$

This verifies that the elements of the 5D metric have transformation properties consistent with our interpretation of them. Under purely 4D transformations, $g_{\mu\nu}$, A^μ, and $\widetilde{g}_{55}$ transform as 4space tensor, vector, and scalar, respectively. Under x^5 transformations, $g_{\mu\nu}$ and $\widetilde{g}_{55}$ are unchanged, but A^μ gains the gradient of a scalar, a quantity which does not affect the fields. This invites the interpretation of varying the vector potential by the gradient of a scalar: it corresponds to a translation in x^5. This would also provide a physical interpretation of the Aharanov-Bohm effect.

5. Review of 4D Field Equations

The field equations for the metric $g_{\mu\nu}$ are formulated in terms of the curvature tensor:

$$R^\alpha_{\beta\gamma\delta} = \partial_\delta \Gamma^\alpha_{\beta\gamma} - \partial_\gamma \Gamma^\alpha_{\beta\delta} + \Gamma^\rho_{\beta\gamma}\Gamma^\alpha_{\rho\delta} - \Gamma^\rho_{\beta\delta}\Gamma^\alpha_{\gamma\rho}$$

successive contractions of which yield the Ricci tensor:

$$R_{\alpha\beta} \equiv R^\gamma_{\alpha\gamma\beta}$$

and the curvature scalar:

$$R = g^{\alpha\beta}R_{\alpha\beta}$$

Then the field equations for $g_{\mu\nu}$ are:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \equiv G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Where $T_{\mu\nu}$ is a stress energy tensor, for which there is no real prescription for calculation. As Einstein said, the terms in $g_{\mu\nu}$ are marble, while the other side of the equation wood.

The electromagnetic field equations are

$$\nabla_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu$$

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

where the divergence is covariant; antisymmetry of $F_{\alpha\beta}$ allows covariant derivatives to simplify to partial derivatives in its definition.

6. 5D Field Equations

A core assumption of the theory developed here is that no field variable depends on x^5

$$\frac{\partial \widetilde{g}_{ab}}{\partial x^5} = 0 \tag{21}$$

The invariance with respect to x^5 is the cylinder condition. Wesson (1999) identifies terms associated with the relaxation of the cylinder condition as matter source terms. This turns the wood into marble, the pursuit of many scientists, Einstein included. Here, the cylinder condition is adopted as an approximation motivated by empirics: we do not perceive variations in x^5, although charge is somehow associated with 'motion' in x^5.

Here ϕ is allowed to vary with x^μ, unlike some treatments of Kaluza theory, includingn Kaluza's. Jordan showed that having ϕ vary was required to avoid an unphysical constraint on the electromagnetic field, a result which leads to an interesting interpretation below.

Allowing for a 5D stress-tensor $\widetilde{T}_{ab}$, the 5D field equations are

$$\widetilde{G}_{ab} \equiv \widetilde{R}_{ab} - \frac{1}{2}\widetilde{g}_{ab}\widetilde{R} = \frac{8\pi G}{c^4}\widetilde{T}_{ab} \tag{35}$$

The 5D field equations have been shown by Thiry (1948) to be:

$$\widetilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{k^2\phi}{2}F_{\mu\alpha}F_{\beta\nu}g^{\alpha\beta} + \phi^{-1/2}\nabla_\mu\nabla_\nu\phi^{1/2} \tag{31}$$

$$\widetilde{R}_{\mu5} = \frac{k\phi^{1/2}}{2}\nabla^\alpha F_{\alpha\mu} + \frac{3k}{2}\partial^\alpha\phi^{1/2}F_{\alpha\mu} \tag{32}$$

$$\widetilde{R}_{55} = \phi^{-1/2}\nabla^\mu\nabla_\mu\phi^{1/2} - \frac{k^2\phi}{4}F_{\alpha\beta}F^{\alpha\beta} \tag{33}$$

$$\widetilde{R} = R - 2\phi^{-1/2}\nabla^\mu\nabla_\mu\phi^{1/2} - \frac{k^2\phi}{4}F_{\alpha\beta}F^{\alpha\beta} \tag{34}$$

so that

$$\begin{aligned}\widetilde{G}_{\mu\nu} = G_{\mu\nu} &+ \frac{k^2\phi}{2}(g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}/4 - F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta}) \\ &- \phi^{-1/2}(\nabla_\mu\nabla_\nu\phi^{1/2} - g_{\mu\nu}\nabla_\alpha\nabla^\alpha\phi^{1/2}) \equiv G_{\mu\nu} - 2\pi k^2\phi T^{EM}_{\mu\nu} - T^{\phi}_{\mu\nu}\end{aligned} \tag{36}$$

For $\widetilde{G}_{\mu 5} = \widetilde{G}_{55} = 0$, Thiry shows

$$\nabla^\mu F_{\mu\nu} = -3\phi^{-1/2}\partial^\mu \phi^{1/2} F_{\mu\nu}$$

$$g^{\alpha\beta}\nabla_\alpha \nabla_\beta \phi^{1/2} = \frac{k^2 \phi^{3/2}}{4} F_{\alpha\beta}F^{\alpha\beta} \tag{38}$$

These last are consistent with Wesson (1.27).

7. k from 4D Einstein-Maxwell Limit

Equation (38) shows that the electromagnetic field is a source for the scalar field. Likewise, the scalar field is a source for the electromagnetic field. There are effective contributions to the vacuum stress energy tensor from the scalar and electromagnetic fields.

The dependence of the ϕ on the electromagnetic field shows that variation of ϕ is necessary to avoid constraining the EM field. As discussed by Bargmann (1957), various authors interpreted the scalar field in in terms of a variable gravitational constant, thus making a correspondence with Brans-Dicke theory.

To make contact with existing theory we must require correspondence between the vacuum EM stress energy and the 4D result: $2\pi k^2\phi = 8\pi G/c^4$, implying:

$$k^2\phi = \frac{4G}{c^4} \qquad \Longrightarrow \qquad k^2 \equiv \frac{4G_0}{c^4}; \qquad \phi \equiv \frac{G}{G_0} \tag{39}$$

where the scalar field ϕ has been interpreted in terms of the gravitational 'constant' as in Brans & Dicke (1961), while recognizing that an alternative would be a varying c.

A variant of Brans-Dicke gravity is embedded in (36) and (38), with the electromagnetic field as the source of ϕ. And unlike Brans-Dicke, the ϕ field impacts the equations of motion.

It is not clear why variation in G should be preferred to variation in c. as both are seen to be conversions from charge-space and time, respectively, into length.Perhaps the scalar field should be seen as variation of the ratio G/c^4.

8. G dependence on EM energy density at cosmological scales

Assuming that ϕ indeed describes the variable gravitational or speed of light constants, (38) can be used to discern the lengthscale of the variation of ϕ.

Indeed, (38) implies that a lengthscale similar to the radius of the universe would characterize variation in ϕ of order unity, using an electromagnetic energy density similar to the observed galactic value of 1 eV/cm^3, and G and c similar to what is observed in the present epoch. This variation is of interest only on cosmological scales, and not of interest for the equations of motion of terrestrial objects. Therefore ϕ can be approximated as a constant in the equations of motion, and the constraint is only on the cosmological electromagnetic energy density.

The constancy of ϕ on terrestrial scales makes the identity exact in (16) between the 5D connection and the 4D equations of motion. This is also one of Kaluza's original simplifying assumptions.

The theory developed below is built on the assumption of no variation in the metric with respect to x^5, and a weak assumption of constant ϕ.

Similarly, (38) implies a lengthscale for variations in G, which is given by

$$l_k \sim (k^2 e_{EM})^{-1/2} \sim c^2/(G e_{EM})^{1/2}$$

Since e_{EM} is quadratic in the field strength, a 10^{12} G magnetic field, typical for neutron stars, would yield a characteristic l_k of about 1 lightyear.

9. G as charge-to-mass ratio; size of kA

Consider the units of G: $l^3 m^{-1} t^{-2}$, while charge $q \sim m^{1/2} l^{3/2} t^{-1}$ and vector potential $A \sim m^{1/2} l^{1/2} t^{-1}$. In this case, $k^2 A^2$ is unitless, and is the dimensionless number that expresses the strength of the coupling corrections to the 4-D theory, GA^2/c^4. According to our conventions, $\widetilde{g}_{ab}$ is unitless, with the coordinates of length units, thus ϕ is unitless. It is a number of size yet to be determined. Under the assumption that ϕ is of order unity, we can verify the magnitude of kA. This assumption is reasonable because we expect the existence of a flat 5D space in whose limit $\phi \to 1$.

The first thing that strikes one is the magnitude of the numbers involved. For a 10^6 G magnetic field with characteristic length 1 km, $A \sim 10^{11}$, while $k^2 \sim G/c^4 \sim 10^{-49}$, so that in this case $kA \sim 10^{-38}$. Even the field for a pulsar, 10^{12} G with a 10 km lengthscale, $kA \sim 10^{-31}$. For any terrestrial A^μ, $kA \ll 1$. The corrections to $g_{\mu\nu}$ are quite small enough as to be unobservable.

Now estimate the magnitude of U^5 from (19).

$$U^5 = c\,\frac{q/m}{2G_0^{1/2}} \tag{40}$$

The original requirement from the equations of motion implies that speed in the 5th dimension is fixed by the particle charge-to-mass ratio. Assume that like rest mass, charge is a rest-frame constant multiplying some scale factor which depends on motion. For elementary particles, $U^5/c \gg 1$. For the electron, for example, $U^5_e/c = 10^{21}$. G sets a characteristic charge-to-mass ratio in this theory, and provides an interesting link between gravity and electrodynamics.

10. 5D Stress Tensor

We can use (37) to normalize the 5D extensions to the 4D stress tensor:

$$\widetilde{T}_{5\mu} \equiv \eta J_\mu; \qquad \eta\frac{8\pi G}{c^4} = k\phi^{1/2}\frac{4\pi}{c} \Longrightarrow \eta = c/\sqrt{G} \tag{41}$$

11. Energy-momentum-charge 5-vector: flat space

From (11) and (19), it is clear that electric charge is identified with motion in the 5th dimension, as energy is associated with 'motion' in time, and momentum with motion in space. The traditional energy-momentum 4-vector is seen to be the 4D projection of an energy-momentum-charge 5-vector.

$$\widetilde{U}^a = \frac{dx^a}{d\widetilde{\tau}} = \frac{d\tau}{d\widetilde{\tau}}\left(\frac{E}{mc}, \frac{\mathbf{p}}{m}, \frac{q}{mck}\right) = \frac{dt}{d\widetilde{\tau}}\left(c, \mathbf{v}, s\dot{h}\right) = c\frac{d\tau}{d\widetilde{\tau}}\left(\gamma, \gamma\beta, \frac{q/m}{2G^{1/2}}\right)$$

We may speculate that electric charge q is identified with motion over time in x_5. The constant that converts charge to length units, as c does for time, is $s = 1/ck$, showing that ultimately G is a conversion unit from x_5 to time.

The invariant length of the 5velocity implies

$$\widetilde{\gamma}^2 = \frac{b^2}{c^2}\left(1 - \beta^2 - \frac{(q/m)^2}{4G}\right)^{-1}$$

12. Constants of Motion

The equations of motion (9) can be written in a simple and completely general form for the covariant component of the 5-velocity $\widetilde{U}_a = \widetilde{g}_{ab}\widetilde{U}^b$. Because the metric commutes with the covariant derivative, the equations of motion imply $U^a\nabla_a U^b = 0 = g_{bc}U^a\nabla_a U^b = U^a\nabla_a U_c$. This and antisymmetry in the connections imply:

$$\frac{d\widetilde{U}_a}{d\widetilde{\tau}} = \frac{1}{2}\widetilde{U}^b\widetilde{U}^c\frac{\partial\widetilde{g}_{bc}}{\partial x^a} \tag{23}$$

This form of the equations of motion is in terms of a normal partial derivative instead of a covariant derivative, and expresses the conservation properties of the metric. There is a conserved quantity associated with each invariant coordinate. This is an expression of Noether's theorem: the invariance of the laws of physics in time/space/rotation manifest as conservation of energy/momentum/angular momentum.

The cylinder condition (21) therefore implies a conserved quantity along 5D worldlines:

$$\widetilde{U}_5 = kA_\nu \widetilde{U}^\nu + \phi \widetilde{U}^5; \qquad \frac{d\widetilde{U}_5}{d\widetilde{\tau}} = 0 \tag{25}$$

equivalent *Wesson 5.20.* Note that it is $\widetilde{U}^5$ which is identified with electric charge in the equations of motion (19), while (25) shows that $\widetilde{U}^5$ alone is not conserved, implying non-conservation of the parameter in the Lorentz force law we call charge.

Now consider the case of the 5D metric independent of time, for which there is the conserved quantity:

$$\begin{aligned}\widetilde{U}_0 &\equiv \widetilde{g}_{0a}\widetilde{U}^a = constant \\ &= (g_{0\mu} + k^2 A_0 A_\mu/\phi)\widetilde{U}^\mu + kA_0\widetilde{U}^5 \\ &= g_{0\mu}\widetilde{U}^\mu + \frac{k\widetilde{U}_5}{\phi}A_0\end{aligned} \tag{29}$$

equivalent to *Wesson 5.39.* This generalizes the 4D energy by including the electrostatic contribution.

The 5-momentum which is conserved when the metric is independent of spatial derivatives j:

$$\widetilde{U}_j = g_{j\mu}U^\mu + \frac{k\widetilde{U}_5}{\phi}A_j = constant \tag{30}$$

This is the usual canonical momentum but with the distinction, as in the energy equation, that the charge in the equations of motion is not the same as the effective conserved charge-like quantities.

13. Equations of motion on non-cosmological scales

If variations in ϕ are ignored, then a relatively local regime of lengthscale variation is assumed, which may be called non-cosmological. On such scales, terms in the the gradient of ϕ are ignored and ϕ is taken to be constant.

We are now in a position to consider the equations of motion (12) and (13). For the 4-velocity equation (12), there are 3 connections of interest in the limit of constant ϕ

$$\widetilde{\Gamma}^{\mu}_{55} = \widetilde{\Gamma}^{5}_{55} = 0 = \widetilde{\Gamma}^{a}_{55}$$

$$\widetilde{\Gamma}^{\mu}_{5\alpha} = \frac{k}{2} g^{\mu\nu} F_{\alpha\nu}$$

$$\widetilde{\Gamma}^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta} + \frac{2G}{c^4} g^{\mu\nu} H_{\alpha\beta\nu}; \qquad H_{\alpha\beta\nu} \equiv A_\alpha F_{\beta\nu} + A_\beta F_{\alpha\nu}$$

There is no contribution from the term in U^5. The 5D connection for the cross term reduces to the usual Lorentz formula. However, there is a modification to the equations of motion. We see an apparent antigravity term of electromagnetic origin that could in principle nullify the standard Einstein term.

The constants of motion provide a more direct approach to the equation for U^5 than the connections above. As shown below, $d\tau/d\widetilde{\tau}$ varies with ϕ, so this quantity is approximately constant, and

$$kA_\nu U^\nu + U^5 = \frac{G}{c^3} m A_\nu U^\nu + q = \text{constant}$$

14. Line Element; $d\tau/d\widetilde{\tau}$

With this constant of the motion in hand, consider the 5D line element

$$\begin{aligned} b^2 d\widetilde{\tau}^2 \equiv \widetilde{g}_{ab} dx^a dx^b &= (g_{\mu\nu} + k^2 A_\mu A_\nu/\phi) dx^\mu dx^\nu + 2kA_\mu dx^\mu dx^5 + \phi (dx^5)^2 \\ &= c^2 d\tau^2 + (\phi^{-1/2} \widetilde{U}_5 d\widetilde{\tau})^2 \end{aligned} \tag{27}$$

This provides a relationship between the 4D and 5D line elements

$$\left(\frac{d\tau}{d\widetilde{\tau}}\right)^2 = \frac{b^2}{c^2} - \frac{\widetilde{U}_5^2}{\phi c^2} \tag{28}$$

equivalent to *Wesson 5.22*, and which apparently has a singularity. The invariant length of the the τ-referenced 5-velocity is

$$\widetilde{g}_{ab} U^a U^b = c^2 + \left(\frac{d\widetilde{\tau}}{d\tau}\right)^2 \frac{\widetilde{U}_5^2}{\phi}$$

15. Motion at Constant Time

For motion at constant time, the momentum 5-vector (?) should be cast in terms of the derivative with respect to h. Then

$$\widetilde{U}^a = \frac{dh}{d\widetilde{\tau}} \left(c\frac{dt}{dh}, \frac{d\mathbf{x}}{dh}, s \right) = \frac{dh}{d\widetilde{\tau}} \left(0, \frac{d\mathbf{x}}{dh}, s \right)$$

The interval at constant time is

$$b^2 d\widetilde{\tau}^2 = \widetilde{g}_{ab} dx^a dx^b = -d\mathbf{x}^2 + \mathrm{e}^{i\theta} s^2 dh^2$$

so that

$$\left(\frac{dh}{d\widetilde{\tau}}\right)^2 = \frac{b^2}{\mathrm{e}^{i\theta} s^2 - (d\mathbf{x}/dh)^2}$$

Also use $\widetilde{T}_{ab} = m\widetilde{U}^a \widetilde{U}^b$.

16. Appendix A: Algebraic Relations

The following useful relations are from Weinberg.

Since the scalar curvature is related to the trace of the stress energy tensor

$$R = \frac{8\pi G}{c^4} g^{\alpha\beta} T_{\alpha\beta}$$

the field equations can be rewritten to show vacuum equations purely in terms of the Ricci tensor:

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_{\alpha\beta} g^{\alpha\beta} \right)$$

The contraction of the curvature tensor with the metric, $g_{\alpha\nu} R^{\nu}_{\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$, has the properties of being symmetric in its pairs of indices

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

antisymmetric to permutations of a pair

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$$

and cyclic in the last three indices

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$$

Some useful algebraic results:

$$\partial_\lambda g_{\mu\nu} = g_{\rho\nu} \Gamma^{\rho}_{\lambda\mu} + g_{\mu\rho} \Gamma^{\rho}_{\lambda\nu}$$

$$R_{\lambda\mu\nu\alpha} = \frac{1}{2} [\partial_\alpha \partial_\mu g_{\lambda\nu} - \partial_\alpha \partial_\lambda g_{\mu\nu} - \partial_\nu \partial_\mu g_{\lambda\alpha} + \partial_\nu \partial_\lambda g_{\mu\alpha}] + g_{\eta\sigma} [\Gamma^{\eta}_{\nu\lambda} \Gamma^{\sigma}_{\mu\alpha} - \Gamma^{\eta}_{\alpha\lambda} \Gamma^{\sigma}_{\mu\nu}]$$

$$R = g^{\lambda\nu} g^{\mu\alpha} R_{\lambda\mu\nu\alpha}$$

$$R_{\alpha\beta} = g^{\lambda\nu} R_{\lambda\alpha\nu\beta}$$

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\nu \sqrt{g}$$

17. Appendix B: Stress Energy Tensor Forms

Electromagnetic stresses are, however, a source of curvature. An (they are not unique) electromagnetic stress-energy tensor is:

$$T_{EM}^{\alpha\beta} = \frac{1}{4\pi}(g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\lambda}F^{\mu\lambda})$$

So

$$T_{EM}^{00} = \frac{1}{8\pi}(E^2 + B^2) \qquad T_{EM}^{oi} = \frac{1}{4\pi}\epsilon_{ijk}E^jB^k$$

$$T_{EM}^{ij} = -\frac{1}{4\pi}\left[E^iE^j + B^iB^j - \frac{1}{2}\delta_{ij}(E^2 + B^2)\right]$$

A useful material stress-energy tensor is that for a fluid:

$$T^{\mu\nu} = Pg^{\mu\nu} + (P + \rho c^2)\frac{U^\mu U^\nu}{c^2}$$

Or for a collection of non-interacting particles:

$$T^{\mu\nu} = \rho U^\mu U^\nu$$

18. Appendix C: 5D Connections

The 5D connections:

$$\widetilde{\Gamma}^5_{55} = \frac{1}{2} k A^\mu \partial_\mu \ln\phi = -\widetilde{\Gamma}^\alpha_{5\alpha} \Longrightarrow \widetilde{\Gamma}^a_{5a} = 0$$

$$\widetilde{\Gamma}^5_{5\mu} = \frac{1}{2}(1 + k^2 A^2/\phi)\partial_\mu \ln\phi - \frac{k^2}{2\phi} A^\nu F_{\mu\nu}$$

$$\widetilde{\Gamma}^\mu_{55} = -\frac{1}{2} g^{\mu\nu} \partial_\nu \phi$$

$$\widetilde{\Gamma}^\mu_{5\nu} = \frac{1}{2} k g^{\mu\sigma} F_{\nu\sigma} - \frac{1}{2} k A^\mu \partial_\nu \ln\phi$$

$$\widetilde{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \frac{k^2}{2\phi} g^{\mu\nu}(A_\alpha F_{\beta\nu} + A_\beta F_{\alpha\nu}) + \frac{k^2}{2\phi} g^{\mu\nu}(A_{\beta\nu}\partial_\alpha \ln\phi + A_{\alpha\nu}\partial_\beta \ln\phi - A_{\alpha\beta}\partial_\nu \ln\phi)$$

so that

$$\widetilde{\Gamma}^a_{\alpha\beta} = \Gamma^\alpha_{\alpha\beta} + \left[\frac{1}{2} + \frac{k^2 A^2}{2\phi} + \frac{k^2 A^2}{2\phi^2}\right] \partial_\beta \ln\phi$$

$$\widetilde{\Gamma}^5_{\alpha\beta} = \frac{k}{2\phi}(H_{\alpha\beta} - 2A_\sigma \Gamma^\sigma_{\alpha\beta}) - \frac{k^3}{2\phi^2} H_{\alpha\beta\nu} A^\nu$$

where

$$A_{\alpha\beta} \equiv \frac{A_\alpha A_\beta}{\phi} \qquad H_{\alpha\beta\nu} \equiv A_\alpha F_{\beta\nu} + A_\beta F_{\alpha\nu}$$

Some traces:

$$\widetilde{\Gamma}^\alpha_{\alpha\beta} = \Gamma^\alpha_{\alpha\beta} + \frac{k^2}{2\phi} A^\alpha F_{\beta\alpha}$$

$$\widetilde{\Gamma}^\alpha_{\alpha 5} = 0$$

A couple useful combinations:

$$\widetilde{\Gamma}^5_{\alpha\beta} + \frac{k A_\mu}{\phi} \widetilde{\Gamma}^\mu_{\alpha\beta} = \frac{k}{2\phi} H_{\alpha\beta}$$

$$\widetilde{\Gamma}^5_{\mu 5} + \frac{k A_\alpha}{\phi} \widetilde{\Gamma}^\alpha_{5\mu} = 0$$

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